THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Problem Set 2

1. Find the following limits.

   (a) $\lim_{n\to\infty} \dfrac{3n^2 - 2n + 7}{2n^2 + 3}$

   (b) $\lim_{n\to\infty} \dfrac{-3n^2}{\sqrt[3]{27n^6 - 5n + 1}}$

   (c) $\lim_{n\to\infty} \sqrt{4n^2 + n} - \sqrt{4n^2 - 1}$

   (d) $\lim_{n\to\infty} \dfrac{\sin(2^n) + (-1)^n \cos(2^n)}{n^3}$ (Hint: Using the sandwich theorem)

2. Let $\{a_n\}$ be a sequence of positive real numbers, which is defined by

$$a_1 = 1 \qquad \text{and} \qquad a_n = \frac{12a_{n-1} + 12}{a_{n-1} + 13} \text{ for } n > 1.$$

   (a) Prove that $a_n \leq 3$.

   (b) Prove that $\{a_n\}$ converges (i.e. $\lim_{n\to\infty} a_n$ exists) and hence find its limit.

3. (a) Prove that $\dfrac{2^n}{n!} \leq \dfrac{4}{n}$ for all natural numbers $n \geq 2$.

   (b) Hence, show that $\lim_{n\to\infty} \dfrac{2^n}{n!} = 0$.

4. By considering $\dfrac{1}{\sqrt{n^2+n}} \leq \dfrac{1}{\sqrt{n^2+r}} \leq \dfrac{1}{\sqrt{n^2+1}}$ for $r = 1, 2, 3, \cdots, n$
   and the sandwich theorem, prove that

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) = 1.$$

5. (Harder problem)

   Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that $x_1 = 2$ and $y_1 = 8$ and for $n = 1, 2, 3, \cdots$

$$x_{n+1} = \frac{x_n^2 y_n + x_n y_n^2}{x_n^2 + y_n^2} \qquad \text{and} \qquad y_{n+1} = \frac{x_n^2 + y_n^2}{x_n + y_n}.$$

   (a) Prove that $x_{n+1} - y_{n+1} = \dfrac{-(x_n^3 - y_n^3)(x_n - y_n)}{(x_n + y_n)(x_n^2 + y_n^2)}$ for all natural numbers n.

   (b) Show that $0 \leq x_n \leq y_n$ for all natural numbers n.
   Hence, prove that $\{x_n\}$ is a monotonic increasing sequence and $\{y_n\}$ is a monotonic decreasing sequence.

   (c) Prove that $\{x_n\}$ and $\{y_n\}$ converge and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

   (d) Prove that $x_n y_n$ is a constant and hence find $\lim_{n\to\infty} x_n$.